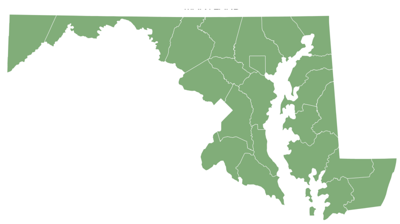
MTH241 Fall 2024: Exam 03

Instructor: Dr. Ling Liang

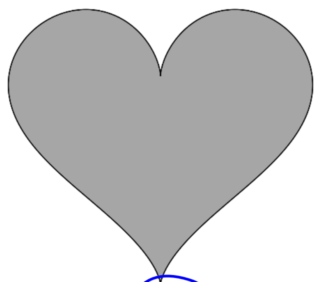
Name:
UID:

Closed book, no calculator, show your work clearly. Read the problems carefully.

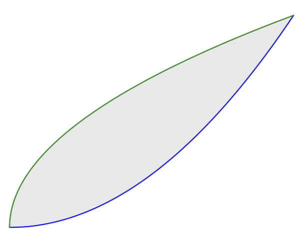
1. (15pt) Determine whether the following regions are vertically simple, or horizontally simple, or both. Check whatever is correct.



Horizontally Simple | Vertically Simple | Simple | None

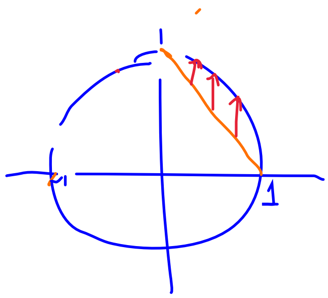


Horizontally Simple | Vertically Simple | Simple



Horizontally Simple | Vertically Simple | Simple

2. (20pt) Let R be the region in the plane between $x^2 + y^2 = 1$ and $y = 1 - x$, and in the first quadrant. Sketch the region R in the xy-plane and write down the iterated double integral for $\iint_R xy dA$, treating R as a vertically simple region. Evaluate the integral.



$$\int_0^1 \int_{y=1-x}^{y=\sqrt{1-x^2}} xy \, dy \, dx$$

$$= \int_0^1 x \frac{y^2}{2} \Big|_{1-x}^{\sqrt{1-x^2}} dx = \int_0^1 \frac{x}{2} \left(1 - x^2 - (1-x)^2\right) dx$$

$$= \int_0^1 \frac{x}{2} \left(2x - 2x^2\right) dx$$

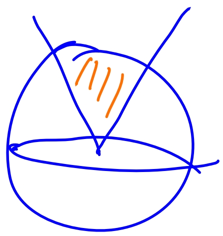
$$= \int_0^1 x^2 - x^3 \, dx = \frac{x^3}{3} - \frac{x^4}{4} \Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

3. (15pt) Let D be the solid region inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{2x^2 + 2y^2}$. Describe D using spherical coordinate, i.e., find $\alpha, \beta, h_1, h_2, F_1, F_2$ such that D is as

$$\alpha \leq \theta \leq \beta, \quad h_1(\theta) \leq \phi \leq h_2(\theta), \quad F_1(\phi, \theta) \leq \rho \leq F_2(\phi, \theta).$$

Moreover, set up the iterated integral for $\iiint_D x^2 dV$ (Don't evaluate the integral).



$$\int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\text{cone}} \int_{\rho=0}^{\rho=2} \rho^2 \cos^2(\theta) \sin^2(\varphi) \cdot \rho^2 \sin(\varphi) \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_{\varphi=0}^{\varphi=\arctan(\frac{\sqrt{2}}{2})} \int_{\rho=0}^{\rho=2} \rho^4 \cos^2(\theta) \sin^3(\varphi)$$

$z = \sqrt{2(x^2+y^2)}$

$\rho\cos(\varphi) = \sqrt{2}\, \rho \sin(\varphi)$

$\Rightarrow \tan(\varphi) = \frac{\sqrt{2}}{2}$

$\Rightarrow \varphi = \arctan(\frac{\sqrt{2}}{2})$

4. (10pt) Perform a change of variable to evaluate the integral $\iint_R 1 dA$ where R is the region inside $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

$x = 3r\cos(\theta)$

$y = 4r\sin(\theta)$

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 3\cos(\theta) & -3r\sin(\theta) \\ 4\sin(\theta) & 4r\cos(\theta) \end{bmatrix} \rightsquigarrow \det(J) = 12r$$

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} 12r \quad dr\, d\theta = \int_0^{2\pi} 6 \, d\theta = 12\pi$$

5. (15pt) Let the smooth curve C be parameterized by $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ for $t \in [0, 2\pi]$. Evaluate the line integral

$$\int_C \frac{z^2}{x^2+y^2}\,ds.$$

$\frac{dr}{dt} = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{bmatrix} \rightarrow \left\|\frac{dr}{dt}\right\| = \sqrt{2}$

$\int_0^{2\pi} f(r(t)) \left\|\frac{dr}{dt}\right\| = \int_0^{2\pi} \frac{t^2}{1} \cdot \sqrt{2}\; dt = \sqrt{2}\,\frac{(2\pi)^3}{3}$

6. Let $\vec{F}(x,y) = -y\vec{i} + x\vec{j} + 0\vec{k}$ be a vector field and C be the circle in the xy-plane centered at $(0,0)$ with radius 1, oriented counterclockwise.

(a) (10pt) Determine whether there exists a function $f(x,y)$ such that grad $f = \vec{F}$.

$F = \begin{bmatrix} -y \\ x \end{bmatrix}$ $\quad M_y = -1 \neq N_x = 1$ $\quad$ so $\underline{\text{No}}$

(b) (15pt) Evaluate the following integral $\int_C \vec{F} \cdot d\vec{r}$.

$\int_C F \cdot \partial r = \int_0^{2\pi} F(r(t)) \cdot \frac{dr}{dt}$ $\quad r(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ $\quad 0 \le t \le 2\pi$

$= \int_0^{2\pi} \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \partial t = \int_0^{2\pi} 1 \, \partial t = 2\pi$

Extra page: